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PROOF OF A THEOREM IN CONTINUED FRACTIONS.

By Mr. Derrick N. Lehmer, Lincoln, Neb.

The theorem in question is : If $\sqrt{R}$ be expressed as a continued fraction ;

$$\sqrt{R} = q_0 + \frac{1}{q_1 +}\,\frac{1}{q_2 +} \dots + \frac{1}{q_n +}\,\frac{1}{q_1 +} \dots$$

then $q_n = 2q_0$. The proofs usually given of this theorem require so many auxiliary propositions that the following proof may be as welcome to others as it was to me.

Let

$$x = \sqrt{R} - q_0 = \frac{1}{q_1 +}\,\frac{1}{q_2 +} \dots + \frac{1}{q_n +}\,\frac{1}{q_1 +} \dots .$$

Denote the nth convergent fraction by A_n/B_n ; then

$$\frac{1}{x} = \frac{A_n/x + A_{n-1}}{B_n/x + B_{n-1}}$$

or

$$x^2 + x(A_n - B_{n-1})/A_{n-1} - B_n/A_{n-1} = 0 .$$

One root of this equation is $\sqrt{R} - q_0$ and since the coefficients are rational the other root must be $-\sqrt{R} - q_0$, and the equation in x is

$$x^2 + 2q_0x - (N - q_0^2) = 0 .$$

Equating coefficients we have $(A_n - B_{n-1})/A_{n-1} = 2q_0$.

Now $A_n = q_n A_{n-1} + A_{n-2}$, whence

$$q_n + (A_{n-2} - B_{n-1})/A_{n-1} = 2q_0 .$$

Unless $(A_{n-2} - B_{n-1})/A_{n-1}$ is an integer the last equation cannot hold, as we would then have an integer equal to a fraction. Since then $A_{n-2} < A_{n-1}$ the fraction must be zero, or $A_{n-2} = B_{n-1}$, which is, I believe, a new proof of the theorem. We have at once then

$$q_n = 2q_0 .$$

October, 1896.